The Numerical Solution of Volterra Integral Equations of the First Kind Using Hermite Polynomials via the Galerkin's Residual Method

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Abstract
In this paper, the method for solving Volterra integral equations of the first kind with weakly singular kernels using the Hermite polynomials is presented. The procedure resulted in the construction of systems of algebraic equations, solving these systems of algebraic equations an approximate solution $\bar{y}(x)$ is obtained numerically; Illustrative examples are included to demonstrate the simplicity and applicability of the method. Once the approximate solution coincides with the exact solution for any particular value of n, further evaluations can only give an approximate solution. The volume of work involve in this method is much easier than most of the existing methods contained in the literature.

Keywords: *Galerkin's method*, *Hermite polynomials*, *kernels, Volterra integral equations*, *kernels*.

1.0 Introduction

It's well known that linear and nonlinear volterra integrals are mathematical models considered by many evolutionary problems which arise in many scientific fields such as the population dynamics, spread of epidemics and semi-conductor devices. Its applications are found in the areas of ruin theory and in the study of the risk of insolvency in actuarial science. The growing interest in this equation arising in various fields such as Physics, Engineering and Economics in recent time's motivated mathematicians to develop reliable methods for solving it. Approximate methods for solving numerically various classes of Volterra integral equations (VIEs) are very rare. Several methods have been proposed for the numerical solutions of these equations. Prior to the development of finite element method, there existed an approximation technique for solving differential and integral equations called the Weighted Residuals Methods (WRMs). The Galerkin's method considered in this work is among the earliest of these methods.

Numerical solutions of VIEs have been studied by many scientist using different approaches and methods. The Numerical Solutions of VIE using Laguerre Polynomials was discussed in [2], the Bernstein polynomials was used in the approximation techniques for VIE in [3, 5, 7]. Taylor series polynomials were used for the numerical solution of VIE in [4, 5, 8] among other methods. Since the piecewise polynomials are differentiable and integrable, the Hermite polynomials is proposed for solving integral equations of the VIE numerically. The method transforms the integral equation to a linear system of algebraic equations which are then solved by direct or iterative methods using the matrices of the linear systems of algebraic equations involve. In most of these methods the matrices involve is always very expensive to obtain computationally. Therefore the introduction of the Galerkin's weighted residual method transforms the VIE to linear systems of algebraic equations whose matrix is very simple to handle using Maple Software and the accuracy of these methods depends on the choice of the trial function.

2.0 Hermite Polynomials

The general form of the Hermite polynomials of n^{th} degree is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}, \qquad n = 0,1,2,3,\ldots, \qquad -\infty < x < \infty \qquad (1)$$

Below are some of the first few Hermite polynomials obtained from recurrence relation (1)

$$H_0(x) = 1, \qquad H_1(x) = 2x, \; H_2(x) = 4x^2 - 2, \qquad H_3(x) = 8x^3 - 12x,$$
$$H_4(x) = 16x^4 - 48x^2 + 12, \; H_5(x) = 35x^5 - 160x^3 + 120x$$
$$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$

3.0 Methodology

An integral equation of the form;

$$\int_{x_0}^{x} \mathcal{L}(x,t) y(t) dt = f(x), \qquad \alpha \le x \le \beta \qquad (2)$$

where $\mathcal{L}(x,t)$ is called the integral kernel (nucleus), $f(x)$ is a specified real valued continuously differentiable function defined on $[\alpha, \beta]$, and $y(x)$ is the function to be solved for is called a Volterra integral equation of the first kind where the unknown function $y(x)$ only appears inside the integral sign. In this paper, $f(x)$ is strictly a polynomial of degree $n \ge 1$, satisfying $f(x_0) = 0$.

In this case, we shall assume an approximate solution given by

$$\bar{y}(x) \cong y(x) = \sum_{i=0}^{n} a_i H_i(x) \qquad (3)$$

where $H_i(x)$ is a Hermite polynomials of degree i defined in (1), n represent the number of Hermite polynomials and $a_i{}'s$ are the unknown constant parameters we need to determine. Now substituting (3) in (2) to obtain;

$$\sum_{i=0}^{n} a_i \int_{x_0}^{x} \mathcal{L}(x,t) H_i(x) dt = f(x), \qquad \alpha \le x \le \beta \qquad (4)$$

In order to obtain the Galerkin's equations, both sides of (4) is multiplied by $H_q(x)$ and integrating the result with respect to x over the interval $[\alpha, \beta]$ to obtain

$$\sum_{i=0}^{n} a_i \int_a^{\beta} \left(\int_{x_0}^{x} \mathcal{L}(x,t) H_i(t) dt \right) H_q(x) dx = \int_a^{\beta} f(x) H_q(x) dx \, , q = 0,1, \dots, n \quad (5)$$

From equation (5) for each $q = 0,1,2, \dots, n$, $(n+1)$, $(n+1)$ linear equations with $(n+1)$ unknowns are obtained and can be put in matrix notation as;

$$\sum_{i=0}^{n} a_i \, \mathcal{M}_{i,q} = \mathcal{F}_q \, , i, q = 0,1,2, \dots., n \quad (6)$$

where $\mathcal{M}_{i,q} = \int_a^{\beta} \left[\int_{x_0}^{x} \mathcal{L}(x,t) H_i(t) dt \right] H_q(x) dx$, $i, q = 0,1,2, \dots, n$ and $\mathcal{F}_q = \int_a^{\beta} f(x) H_q(x) dx$, $q = 0,1,2, \dots, n$

4.0 Numerical Illustrations

In this section, we illustrate the above mentioned method with the help of the following two numerical examples of the first kind with regular kernels and weakly singular kernels, available in some existing literature [2, 10]. The computations, associated with the examples, are performed using the Maple 18 Software.

Example 1

Consider an Abel's integral equation (VIE of first kind with weakly singular kernels) of the form [2]

$$\int_0^x \frac{1}{\sqrt{(x-t)}} y(t) dt = \frac{2}{105} \sqrt{x} (105 - 56x^2 + 48x^3), \quad 0 \le x \le 1$$

The exact solution $y(x) = x^3 - x^2 + 1$ using the method illustrated in section (3.0), solving the linear system with $n = 1, \dots, 5$ we obtain the approximate solutions for each value n and their corresponding absolute errors as shown in Tables 1 and 2 respectively.

Example 2

Consider the Volterra integral equation of the first kind [10] given by,

$$\int_0^x e^{-x+t} y(t) dt = x, \quad 0 \le x \le 1$$

Which has the exact solution $y(x) = x + 1$, using the method illustrated in section (3.0), solving the linear system with $n = 1, \dots, 5$, we obtain the approximate solution for each value n and their corresponding absolute errors as shown in Tables 3 and 4 respectively.

Table 1: Approximate solution for problem 1 for each value n

$n = 1,\ \bar{y}(x) = 0.98846 - 0.16162x$
$n = 2,\ \bar{y}(x) = 0.38462x^2 - 0.5035x + 1.0373$
$n = 3,\ \bar{y}(x) = x^3 - x^2 + 1 \quad \leftarrow$ **(Exact)**
$n = 4,\ \bar{y}(x) = -2.935 \times 10^{-23}x^4 + x^3 - x^2 + 1$
$n = 5,\ \bar{y}(x) = 9.1221 \times 10^{-30}x^5 - 9.8813 \times 10^{-25}x^4 + x^3 - 1.0\,x^2 + 3.1276 \times 10^{-29}x + 1.0$

Table 2: Approximate solution for problem 1 for each value n

n	x	Exact $y(x)$	Approx. $\bar{y}(x)$	Absolute Error
1	0.0	1.0000000	1.0000000	0.0115400
	0.2	1.2000000	0.956136	0.0118640
	0.4	1.4000000	0.923812	0.0198120
	0.6	1.6000000	0.891488	0.0354880
	0.8	1.8000000	0.859164	0.0128360
	1.0	1.0000000	0.826840	0.1731600
2	0.0	1.0000000	1.0373000	0.0373000
	0.2	0.9680000	0.9519848	0.0160152
	0.4	0.9040000	0.8974392	0.0065608
	0.6	0.8560000	0.8736632	0.0176632
	0.8	0.8720000	0.8806568	0.0086568
	1.0	1.0000000	0.9184200	0.0815800
3	0.0	1.0000000	1.0000000	0.0000000
	0.2	0.9680000	0.9680000	0.0000000
	0.4	0.9040000	0.9040000	0.0000000
	0.6	0.8560000	0.8560000	0.0000000
	0.8	0.8720000	0.8720000	0.0000000
	1.0	1.0000000	1.0000000	0.0000000
4	0.0	1.0000000	1.0000000	0.0000000
	0.2	0.9680000	0.9680000	0.0000000
	0.4	0.9040000	0.9040000	0.0000000
	0.6	0.8560000	0.8560000	0.0000000
	0.8	0.8720000	0.8720000	0.0000000
	1.0	1.0000000	1.0000000	0.0000000
5	0.0	1.0000000	1.0000000	0.0000000
	0.2	0.9680000	0.9680000	0.0000000
	0.4	0.9040000	0.9040000	0.0000000
	0.6	0.8560000	0.8560000	0.0000000
	0.8	0.8720000	0.8720000	0.0000000
	1.0	1.0000000	1.0000000	0.0000000

Table 3: Approximate solution for problem 2 for various values of n

$n = 1,\ \bar{y}(x) = 1.0 + x$ ← **(Exact)**

$n = 2,\ \bar{y}(x) = 5.1368 \times 10^{-27}x^2 + x + 1.0$

$n = 3,\ \bar{y}(x) = 1.9875 \times 10^{-24}x^3 - 2.5965 \times 10^{-24}x^2 + x + 1.0$

$n = 4,\ \bar{y}(x) = 9.6656 \times 10^{-24}x^4 - 1.494 \times 10^{-23}x^3 + 6.7288 \times 10^{-24}x^2 + x + 1.0$

$n = 5,\ \bar{y}(x) = -7.7627 \times 10^{-29}x^5 + 1.1481 \times 10^{-23}x^4 - 1.8564 \times 10^{-23}x^3 + 9.0637 \times 10^{-24}x^2 + x + 1.0$

Table 4: Approximate solution for problem 2 for various values of n

n	x	Exact $y(x)$	Approx. $\bar{y}(x)$	Absolute Error
1	0.0	1.0000000	1.0000000	0.0000000
	0.2	1.2000000	1.2000000	0.0000000
	0.4	1.4000000	1.4000000	0.0000000
	0.6	1.6000000	1.6000000	0.0000000
	0.8	1.8000000	1.8000000	0.0000000
	1.0	2.0000000	2.0000000	0.0000000
2	0.0	1.0000000	1.0000000	0.0000000
	0.2	1.2000000	1.2000000	0.0000000
	0.4	1.4000000	1.4000000	0.0000000
	0.6	1.6000000	1.6000000	0.0000000
	0.8	1.8000000	1.8000000	0.0000000
	1.0	2.0000000	2.0000000	0.0000000
3	0.0	1.0000000	1.0000000	0.0000000
	0.2	1.2000000	1.2000000	0.0000000
	0.4	1.4000000	1.4000000	0.0000000
	0.6	1.6000000	1.6000000	0.0000000
	0.8	1.8000000	1.8000000	0.0000000
	1.0	2.0000000	2.0000000	0.0000000
4	0.0	1.0000000	1.0000000	0.0000000
	0.2	1.2000000	1.2000000	0.0000000
	0.4	1.4000000	1.4000000	0.0000000
	0.6	1.6000000	1.6000000	0.0000000
	0.8	1.8000000	1.8000000	0.0000000
	1.0	2.0000000	2.0000000	0.0000000
5	0.0	1.0000000	1.0000000	0.0000000
	0.2	1.2000000	1.2000000	0.0000000
	0.4	1.4000000	1.4000000	0.0000000
	0.6	1.6000000	1.6000000	0.0000000
	0.8	1.8000000	1.8000000	0.0000000
	1.0	2.0000000	2.0000000	0.0000000

5.0 Conclusion

In this paper, Volterra integral equations of the first kind are solved using residual method alongside Hermite polynomial basis function. In this method, we obtain the exact solution at one particular value of n and all the remaining values of n, gives only an approximate solution. Thus, the numerical results obtained from the examples demonstrated the validity and efficiency of the proposed method. The authors' conclude that the approximation solutions usually coincide with the exact solutions after the use of some few terms of the polynomial basis function.

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References

[1] J. E. Cicelia, (2014): Solution of Weighted Residual Problems by using Galerkin's Method, *Indian Journal of Science and Technology*, 7(3S), 52–54.

[2] Rahman, M. A., Islam, M. S., and Alam, M. M., (2012): Numerical Solutions of Volterra Integral Equations Using Laguerre Polynomials. *Journal of Scientific research*, 4 (2), 357-364

[3] A. Altürk (2016): Application of the Bernstein Polynomials for Solving Volterra Integral Equations with Convolution Kernels, *Filomat*, 30(4): 1045-1052, DOI: 10.2298/FIL1604045A

[4] A. Bellour and E. A. Rawashdeh (2010): Numerical Solution of First Kind Integral equations by Using Taylor Polynomials, *Journal of Inequalities and Special Functions*, 1(2), 23-29.

[5] K. Wang and Q. Wang (2014): Taylor polynomial method and error estimation for a kind of mixed Volterra-Fredholm integral equations, *Applied Mathematics and Computation*, vol. 229, No. C

[6] Reinkenhof, J., (1977): Differentiation and integration using Bernstein's polynomials, *International Journal of Numerical Methods and Engineering*, **11**, 1627 – 1630.

[7] Kreyszig, E., (1979): Bernstein polynomials and numerical integration, *International Journal of Numerical Methods and Engineering*, **14**, 292 – 295.

[8] Mandal, B. N., Bhattacharya, S., (2007): Numerical solution of some classes of integral equations using Bernstein polynomials, *Applied Mathematics and Computation*, **190**, 1707 – 1716.

[9] Subhra Bhattacharya and Mandal, B.N., (2008): Use of Bernstein Polynomials in numerical solution of Volterra integral equations *Applied Mathematical Sciences*, Vol. 2, 36, 1773 – 1787.

[10] Farshid Mirzaee., (2012): Numerical Solution for Volterra Integral Equations of the First Kind via Quadrature Rule, *Applied Mathematical Sciences*, Vol. 6, **(20),** 969 – 974.